

NOTE ON RECURRENENTS RESOLVABLE INTO A SEQUENCE OF ODD INTEGERS.

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(1) In an examination of the properties of the series

$$1, 3, 10, 35, 126, \dots,$$

the r^{th} term of which is $C_{2r-1, r-1}$, a determinant made its appearance which for the 4th order is

$$\begin{vmatrix} 1 & 1 & . & . \\ 2 \cdot 3_1 & 1 & 1 & . \\ 4 \cdot 2 \cdot 5_2 & 4 \cdot 3_1 & 1 & 1 \\ 6 \cdot 4 \cdot 2 \cdot 7_3 & 6 \cdot 4 \cdot 5_2 & 6 \cdot 3_1 & 1 \end{vmatrix}$$

and has the value $-9 \cdot 11 \cdot 13$. The fact that the value was the product of a sequence of odd integers led to the discovery that under another form it had in effect been already studied. The alternative form in the case of the 4th order is

$$\begin{vmatrix} 1 & 2 & . & . \\ 3_1 & 1 & 4 & . \\ 5_2 & 3_1 & 1 & 6 \\ 7_3 & 5_2 & 3_1 & 1 \end{vmatrix}$$

and is clearly the more convenient of the two. It was brought forward in 1892 in a paper* connected with the theory of integers, where it had for a companion the determinant

$$\begin{vmatrix} 1 & -2 & . & . \\ 3_1 & 1 & -4 & . \\ 5_2 & 3_1 & 1 & -6 \\ 7_3 & 5_2 & 3_1 & 1 \end{vmatrix}$$

These for the n^{th} order, which we may denote by R_n and R'_n , their discoverer had quite skilfully evaluated, his results being

$$\begin{aligned} R_n &= (-1)^{n-1} (2n+1) (2n+3) \dots (4n-3), \\ R'_n &= (2n+3) (2n+5) \dots (4n-1). \end{aligned}$$

(2) The main object of the present paper is to establish a pair of much more general equalities, and to do so not by a mere extension of Reich's

* Reich, K., "Zur Theorie der quadratischen Reste," *Archiv d. Math. u. Physik.* (2), xi, pp. 176-193.

method, but by a quite different and more rapidly effective procedure. Before entering on this, however, we shall, in introducing four fresh results, establish them by the old method, in order that a knowledge of both may be available.

(3) Increasing the last row of R_n , namely,

$$(2n-1)_{n-1}, (2n-3)_{n-2}, \dots, 5_2, 3_1, 1,$$

by

$$\text{row}_{n-1} \cdot 2 + \text{row}_{n-2} \cdot 2 + \text{row}_{n-3} \cdot 4 + \dots,$$

where the r^{th} of the multipliers

$$2, 2, 4, 10, 28, \dots$$

is $\frac{4}{r} C_{2r-3, r-2}$, we find that the new n^{th} row is divisible by $4n-3$, and that this factor being removed the row becomes

$$\frac{2}{n}(2n-3)_{n-2}, \frac{2}{n-1}(2n-5)_{n-3}, \dots, \frac{2}{3} \cdot 3_1, \frac{2}{2} \cdot 3_0, 1.$$

It thus follows that, if we denote the resulting determinant by U_n , we have

$$U_n = \frac{R_n}{4n-3} = (-1)^{n-1} (2n+1) (2n+3) \dots (4n-5); \quad \text{(I)}$$

for example,

$$U_4 = \begin{vmatrix} 1 & 2 & \cdot & \cdot \\ 3 & 1 & 4 & \cdot \\ 10 & 3 & 1 & 6 \\ 5 & 2 & 1 & 1 \end{vmatrix} = -9 \cdot 11.$$

Note should be taken of the relation between the the series of multipliers and the row which ultimately comes of using them, namely, that *the multipliers when halved give the elements of the said n^{th} row in reverse order.*

(4) Again, by adding to 3 times the n^{th} row of R_n

$$1 \cdot \text{row}_{n-1} + 2 \cdot \text{row}_{n-2} + 5 \cdot \text{row}_{n-3} + \dots$$

we obtain a new n^{th} row which is divisible by $2n+1$ and which on removal of the said factor becomes

$$\frac{2}{n+1}(2n-1)_{n-1}, \frac{2}{n}(2n-3)_{n-2}, \dots, \frac{2}{4} \cdot 5_2, \frac{2}{3} \cdot 3_1, \frac{2}{2} \cdot 1_0;$$

so that, if the resulting determinant be denoted by V_n , we have

$$V_n = (-1)^{n-1} 3 (2n+3) (2n+5) \dots (4n-3) \quad \text{(II)}$$

Here the multipliers are

$$1, 2, 5, 14, \dots, \frac{2}{n} (2n-3)_{n-2}$$

and the last row of V_n is

$$\frac{2}{n+1} (2n-1)_{n-1}, \dots, 14, 5, 2, 1.$$

(9) Let us now consider a determinant differing from R_n in the last row, which instead of being

$$(2n-1)_{n-1}, (2n-3)_{n-2}, \dots, 5_2, 3_1, 1$$

is

$$(2n+m-4)_{n-1}, (2n+m-6)_{n-2}, \dots, (m+2)_2, m_1, 1.$$

This determinant may appropriately be denoted by $R_n(m)$, which implies, of course, that what we have hitherto called R_n would now be denoted by $R_n(3)$.

On account of the equality

$$r_s = (r-1)_s + (r-1)_{s-1}$$

the last row of $R_n(m)$ may be partitioned into two, namely, into

$$+ \begin{array}{l} (2n+m-5)_{n-1}, (2n+m-7)_{n-2}, \dots, (m+1)_2, (m-1)_1, 1, \\ (2n+m-5)_{n-2}, (2n+m-7)_{n-3}, \dots, (m+1)_1, \quad 1, \quad 0; \end{array}$$

so that we have at once the recurrent law of formation

$$R_n(m) = R_n(m-1) - 2(n-1) R_{n-1}(m+1) \quad \dots \quad \text{(VI)}$$

If now we view $R_n(3)$ as known, namely,

$$R_n(3) = (-1)^{n-1} (2n+1) (2n+3) \dots (4n-3)$$

we can readily evaluate $R_n(m)$. For, when m is 1 the n^{th} row of the determinant is identical with the $(n-1)^{\text{th}}$ row save in the n^{th} place, so that we have

$$R_n(1) = - (2n-3) R_{n-1}(3)$$

whence by substitution

$$R_n(1) = (-1)^{n-1} (2n-3) (2n-1) \dots (4n-7);$$

and, in the next place, putting $m=2$ in (VI) we have

$$R_n(2) = R_n(1) - 2(n-1) R_{n-1}(3)$$

whence by substitution

$$R_n(2) = (-1)^{n-1} (2n-1) (2n+1) \dots (4n-5).$$

We thus have a formula for $R_n(m)$ which holds for three consecutive values of m ; and, this being the case, the recurrence formula (VI) enables us to show that it holds generally, namely,

$$R_n(m) = (-1)^{n-1} (2n+2m-5) (2n+2m-3) \dots (4n+2m-9) \quad \dots \quad \text{(VII)}$$

(10) It is of importance, however, not to assume a knowledge of the value of $R_n(3)$, but to establish another recurrence-formula which is effective without it, namely,

$$R_n(m) = - (2n+2m-5) R_{n-1}(m+2) \quad \dots \quad \text{(VIII)}$$

This is done by performing on $R_n(m)$ the operation

$$(2n-2) \cdot \text{row}_n - \text{row}_{n-1} \\ - (m-1) \left\{ \text{row}_{n-2} + \frac{1}{2} (m+2)_1 \cdot \text{row}_{n-3} + \frac{1}{3} (m+4)_2 \cdot \text{row}_{n-4} + \dots \right\}$$

$R_n(m+1) = (-1)^{n-1}(2n+2m-3)(2n+2m-5) \dots (4n+2m-7)$
it follows with the help of (XI) that

$$R_n(m+1) = (-1)^{n-1}R'_n(m)$$

i.e., when n is 5

$$\begin{vmatrix} 1 & 2 & . & . & . \\ 3 & 1 & 4 & . & . \\ 10 & 3 & 1 & 6 & . \\ 35 & 10 & 3 & 1 & 8 \\ (m+7)_4 & (m+5)_3 & (m+3)_2 & (m+1)_1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & . & . & . \\ 3 & 1 & -4 & . & . \\ 10 & 3 & 1 & -6 & . \\ 35 & 10 & 3 & 1 & -8 \\ (m+6)_4 & (m+4)_3 & (m+2)_2 & m_1 & 1 \end{vmatrix}$$

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